Topology and its Applications 157 (2010) 2778–2785

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Generalized contractions on partial metric spaces

Ishak Altun *, Ferhan Sola, Hakan Simsek

Department of Mathematics, Faculty of Science and Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey

ARTICLE INFO

Article history:
Received 13 May 2010
Received in revised form 10 August 2010
Accepted 18 August 2010

Keywords:
Fixed point
Generalized contraction
Partial metric space

ABSTRACT

In the present paper, we give some fixed point theorems for generalized contractive type mappings on partial metric space. Also, a homotopy result is given.

1. Introduction and preliminaries

Fixed point theory became one of the most interesting area of research in the last fifty years. Many authors studied contractive type mappings on a complete metric space X which are generalizations of the following well-known Banach contraction principles: Let (X,d) be a complete metric space and let $F : X \to X$ be a mapping. If there exists $\lambda \in [0,1)$ such that

$$d(Fx, Fy) \leqslant \lambda d(x, y) \tag{1.1}$$

for all $x, y \in X$, then F has a unique fixed point in X. Kannan [8] introduced the contractive condition: there exists $\lambda \in (0, 1/2)$ such that

$$d(Fx, Fy) \leqslant \lambda\big[d(x, Fx) + d(y, Fy)\big] \tag{1.2}$$

for all $x, y \in X$, and proved a fixed point theorem using (1.2) instead of (1.1). The conditions (1.1) and (1.2) are independent, as it was shown by two examples in [9]. In 1971, Reich [14] generalized Banach and Kannan fixed point theorems using the contractive condition: for all $x, y \in X$,

$$d(Fx, Fy) \leqslant \alpha d(x, y) + \beta d(x, Fx) + \gamma d(y, Fy),$$

where α, β, γ are nonnegative reals with $\alpha + \beta + \gamma < 1$. Again in 1971, Ćirić [4] generalized such theorems using the contractive condition: for each $x, y \in X$,

$$d(Fx, Fy) \leqslant \alpha(x, y)d(x, y) + \beta(x, y)d(x, Fx) + \gamma(x, y)d(y, Fy) + \delta(x, y)\big[d(x, Fy) + d(y, Fx)\big], \tag{1.3}$$

where $\alpha, \beta, \gamma, \delta$ are functions from X^2 into $[0, 1)$ such that

$$\lambda = \sup\big\{\alpha(x, y) + \beta(x, y) + \gamma(x, y) + 2\delta(x, y):\ x, y \in X\big\} < 1. \tag{1.4}$$

* Corresponding author.
E-mail address: ishakaltun@yahoo.com (I. Altun).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.08.017

Mappings which satisfying (1.3) and (1.4) Ćirić [4] called generalized contractions. As observed in Ćirić [3], a self-mapping F on a metric space (X,d) is generalized contraction if and only if F satisfies the following condition:

$$d(Fx,Fy) \leqslant \lambda m(x,y),$$

where

$$m(x,y) = \max\left\{d(x,y), d(x,Fx), d(y,Fy), \frac{1}{2}\left[d(x,Fy)+d(y,Fx)\right]\right\},$$

$\lambda \in (0,1)$ and $x, y \in X$. Other generalization of the contractive condition (1.1) is nonlinear type [2], that is, for all $x, y \in X$,

$$d(Fx,Fy) \leqslant \phi\big(d(x,y)\big),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+$ is nonnegative reals) is upper semicontinuous function from the right (i.e. $\alpha_n \to \alpha$ implies $\limsup_{n\to\infty} \phi(\alpha_n) \leqslant \phi(\alpha)$) such that $\phi(t) < t$ for each $t > 0$. Again Matkowski [10] used different type nonlinear contractive condition such that: for all $x, y \in X$,

$$d(Fx,Fy) \leqslant \phi\big(d(x,y)\big),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing and satisfies $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t > 0$. After than in [1] and some other papers, the following condition, which is known as generalized nonlinear contractive condition, was used: for all $x, y \in X$,

$$d(Fx,Fy) \leqslant \phi\big(m(x,y)\big),$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing function such that $\phi(t) < t$ for each $t > 0$.

The aim of this paper is to prove some generalized versions of the result of Matthews using different type condition in partial metric spaces. First, we recall some definitions of partial metric space and some properties of theirs [11–13,15,16].

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(p_2) $p(x,x) \leqslant p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \leqslant p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [5,11].

There are some generalizations of partial metrics. For example, O'Neill [13] generalized it a bit further by admitting negative distances. The partial metric of O'Neill sense is called dualistic partial metric. Also, Heckmann [7] generalized it by omitting small self-distance axiom $p(x,x) \leqslant p(x,y)$. The partial metric of Heckmann sense is called weak partial metric. The inequality $2p(x,y) \geqslant p(x,x) + p(y,y)$ is satisfied for all x, y in a weak partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\varepsilon)\colon x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X\colon p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Let (X,p) be a partial metric space. Then:

A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n,x_m)$.

A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

It is easy to see that, every closed subset of a complete partial metric space is complete.

Lemma 1. *([11,12]) Let (X,p) be a partial metric space.*

(a) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n,x) = 0$ if and only if*

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Lemma 2. *Let (X, p) be a partial metric space, $A \subset X$ and $x_0 \in X$. Define $p(x_0, A) = \inf\{p(x_0, x)\colon x \in A\}$. Then $a \in \overline{A} \Leftrightarrow p(a, A) = p(a, a)$.*

Proof.

$$\begin{aligned}
a \in \overline{A} \quad &\Leftrightarrow \quad B_p(a, \varepsilon) \cap A \neq \emptyset \quad \text{for all } \varepsilon > 0 \\
&\Leftrightarrow \quad p(a, x) < \varepsilon + p(a, a) \quad \text{for all } \varepsilon > 0 \text{ and some } x \in A \\
&\Leftrightarrow \quad p(a, x) - p(a, a) < \varepsilon \quad \text{for all } \varepsilon > 0 \text{ and some } x \in A \\
&\Leftrightarrow \quad \inf\bigl\{p(a, x) - p(a, a)\colon x \in A\bigr\} = 0 \\
&\Leftrightarrow \quad \inf\bigl\{p(a, x)\colon x \in A\bigr\} - p(a, a) = 0 \\
&\Leftrightarrow \quad \inf\bigl\{p(a, x)\colon x \in A\bigr\} = p(a, a) \\
&\Leftrightarrow \quad p(a, A) = p(a, a). \qquad \square
\end{aligned}$$

After the definition of the concept of partial metric space, Matthews [11] obtained a Banach type fixed point theorem on complete partial metric spaces. In Section 2, we give some generalized versions of the fixed point theorem of Matthews. In Section 3, a homotopy result is presented.

2. Main result

We begin the generalized nonlinear contractive type fixed point theorem.

Theorem 1. *Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant \phi\left(\max\left\{p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}\bigl[p(x, Fy) + p(y, Fx)\bigr]\right\}\right) \tag{2.1}$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing function such that $\phi(t) < t$ for each $t > 0$. Then F has a unique fixed point.

Proof. From the conditions on ϕ, it is clear that $\lim_{n\to\infty} \phi^n(t) = 0$ for $t > 0$. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = Fx_{n-1}$ for $n = 1, 2, \ldots.$ Now if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \ldots,$ then it is clear that x_{n_0} is a fixed point of F. Now assume $x_n \neq x_{n+1}$ for all n. Then we have from (2.1)

$$\begin{aligned}
p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\
&\leqslant \phi\left(\max\left\{p(x_n, x_{n-1}), p(x_n, Fx_n), p(x_{n-1}, Fx_{n-1}), \frac{1}{2}\bigl[p(x_n, Fx_{n-1}) + p(x_{n-1}, Fx_n)\bigr]\right\}\right) \\
&\leqslant \phi\left(\max\left\{p(x_n, x_{n-1}), p(x_n, x_{n+1}), \frac{1}{2}\bigl[p(x_{n-1}, x_n) + p(x_n, x_{n+1})\bigr]\right\}\right) \\
&= \phi\bigl(\max\bigl\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\bigr\}\bigr),
\end{aligned} \tag{2.2}$$

since

$$p(x_n, x_n) + p(x_{n-1}, x_{n+1}) \leqslant p(x_{n-1}, x_n) + p(x_n, x_{n+1})$$

and ϕ is nondecreasing. Now if

$$\max\bigl\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\bigr\} = p(x_n, x_{n+1})$$

for some n, then from (2.2) we have

$$p(x_{n+1}, x_n) \leqslant \phi\bigl(p(x_n, x_{n+1})\bigr) < p(x_{n+1}, x_n)$$

which is a contradiction since $p(x_n, x_{n+1}) > 0$. Thus

$$\max\bigl\{p(x_n, x_{n-1}), p(x_n, x_{n+1})\bigr\} = p(x_n, x_{n-1})$$

for all n. Then from (2.2) we have

$$p(x_{n+1}, x_n) \leqslant \phi\bigl(p(x_n, x_{n-1})\bigr)$$

and hence

$$p(x_{n+1}, x_n) \leqslant \phi^n\big(p(x_1, x_0)\big). \tag{2.3}$$

On the other hand, since

$$\max\big\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\big\} \leqslant p(x_n, x_{n+1}),$$

then from (2.3) we have

$$\max\big\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\big\} \leqslant \phi^n\big(p(x_1, x_0)\big). \tag{2.4}$$

Therefore

$$\begin{aligned} p^s(x_n, x_{n+1}) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leqslant 2p(x_n, x_{n+1}) + p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \\ &\leqslant 4\phi^n\big(p(x_1, x_0)\big). \end{aligned}$$

This shows that $\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 0$. Now we have

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leqslant p^s(x_{n+k}, x_{n+k-1}) + \cdots + p^s(x_{n+1}, x_n) \\ &\leqslant 4\phi^{n+k-1}\big(p(x_1, x_0)\big) + \cdots + 4\phi^n\big(p(x_1, x_0)\big). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete then from Lemma 1 (X, p^s) is complete and so the sequence $\{x_n\}$ is converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, x) = 0$. Again from Lemma 1, we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.5}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim_{n,m\to\infty} p^s(x_n, x_m) = 0$ and from (2.4) we have $\lim_{n\to\infty} p(x_n, x_n) = 0$, thus from the definition p^s we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore from (2.5) we have $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m) = 0$. Now we show that $p(x, Fx) = 0$. Assume this is not true, then from (2.1) we obtain

$$\begin{aligned} p(x, Fx) &\leqslant p(x, Fx_n) + p(Fx_n, Fx) - p(Fx_n, Fx_n) \\ &\leqslant p(x, x_{n+1}) + p(Fx_n, Fx) \\ &\leqslant p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}\big[p(x, x_{n+1}) + p(x_n, Fx)\big]\right\}\right) \\ &\leqslant p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}\big[p(x, x_{n+1}) + p(x_n, x) + p(x, Fx) - p(x, x)\big]\right\}\right) \\ &= p(x, x_{n+1}) + \phi\left(\max\left\{p(x, x_n), p(x, Fx), p(x_n, x_{n+1}), \frac{1}{2}\big[p(x, x_{n+1}) + p(x_n, x) + p(x, Fx)\big]\right\}\right) \end{aligned}$$

using the continuity of ϕ and letting $n \to \infty$, we have

$$p(x, Fx) \leqslant \phi\big(p(x, Fx)\big),$$

which is a contradiction. Thus $p(x, Fx) = 0$ and so $x = Fx$. Now let z is another fixed point of F, that is $x \neq z$ then from (2.1), since $p(x, x) = 0$, we have

$$\begin{aligned} p(x, z) &= p(Fx, Fz) \\ &\leqslant \phi\left(\max\left\{p(x, z), p(x, Fx), p(z, Fz), \frac{1}{2}\big[p(x, Fz) + p(z, Fx)\big]\right\}\right) \\ &= \phi\left(\max\left\{p(x, z), p(x, x), p(z, z), \frac{1}{2}\big[p(x, z) + p(z, x)\big]\right\}\right) \\ &= \phi\big(\max\big\{p(x, z), p(z, z)\big\}\big) \\ &= \phi\big(p(x, z)\big), \end{aligned}$$

which is a contradiction. Thus $x = z$. $\square$

Example 1. Let $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$, then it is clear that (X, p) is a complete partial metric space. Let $T : X \to X$, $Fx = \frac{x^2}{1+x}$ for all $x \in X$ and $\phi : [0, \infty) \to [0, \infty)$, $\phi(t) = \frac{t^2}{1+t}$. Then for all $x, y \in X$ with $x \geqslant y$ we have

$$\begin{aligned}
p(Fx, Fy) &= \max\left\{\frac{x^2}{1+x}, \frac{y^2}{1+y}\right\} \\
&= \frac{x^2}{1+x} \\
&= \phi\big(p(x, y)\big) \\
&\leqslant \phi\left(\max\left\{p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}\big[p(x, Fy) + p(y, Fx)\big]\right\}\right).
\end{aligned}$$

This shows that all conditions of Theorem 1 are satisfied and so F has a fixed point in X. But we cannot apply the result of Matthews (see Corollary 2 below) to this example, because there is no $\alpha \in [0, 1)$ such that $p(Fx, Fy) \leqslant \alpha p(x, y)$ for all $x, y \in X$.

If we take $\phi(t) = \lambda t$ for $\lambda \in [0, 1)$ in Theorem 1, we have the following corollary.

Corollary 1. *Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant \lambda \max\left\{p(x, y), p(x, Fx), p(y, Fy), \frac{1}{2}\big[p(x, Fy) + p(y, Fx)\big]\right\}$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then F has a unique fixed point.

Now we state Hardy and Rogers type [6] fixed point theorem.

Theorem 2. *Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant ap(x, y) + bp(x, Fx) + cp(y, Fy) + dp(x, Fy) + ep(y, Fx) \tag{2.6}$$

for all $x, y \in X$, where $a, b, c, d, e \geqslant 0$ and, if $d \geqslant e$, then $a + b + c + d + e < 1$, if $d < e$, then $a + b + c + d + 2e < 1$. Then F has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Define a sequence $\{x_n\}$ in X by $x_n = Fx_{n-1}$ for $n = 1, 2, \ldots$. Now if $x_{n_0} = x_{n_0+1}$ for some $n_0 = 0, 1, 2, \ldots$, then it is clear that x_{n_0} is a fixed point of F. Now assume $x_n \neq x_{n+1}$ for all n. Then we have from (2.6)

$$\begin{aligned}
p(x_{n+1}, x_n) &= p(Fx_n, Fx_{n-1}) \\
&\leqslant ap(x_n, x_{n-1}) + bp(x_n, Fx_n) + cp(x_{n-1}, Fx_{n-1}) + dp(x_n, Fx_{n-1}) + ep(x_{n-1}, Fx_n) \\
&= ap(x_n, x_{n-1}) + bp(x_n, x_{n+1}) + cp(x_{n-1}, x_n) + dp(x_n, x_n) + ep(x_{n-1}, x_{n+1}) \\
&\leqslant (a + c + e)p(x_n, x_{n-1}) + (b + e)p(x_n, x_{n+1}) + (d - e)p(x_n, x_n).
\end{aligned} \tag{2.7}$$

Now if $d \geqslant e$, then from (2.7) we have

$$p(x_{n+1}, x_n) \leqslant \max\left\{\frac{a + c + d}{1 - b - e}, \frac{a + c + e}{1 - b - d}\right\} p(x_n, x_{n-1}) \tag{2.8}$$

for all n. If $d < e$, then from (2.7) by omitting the term $-ed(x_n, x_n)$, we have

$$p(x_{n+1}, x_n) \leqslant \max\left\{\frac{a + c + d + e}{1 - b - e}, \frac{a + c + e}{1 - b - d - e}\right\} p(x_n, x_{n-1}). \tag{2.9}$$

Hence from (2.8) and (2.9) we have

$$p(x_{n+1}, x_n) \leqslant \lambda^n p(x_1, x_0),$$

where

$$\lambda = \begin{cases} \max\{\frac{a+c+d}{1-b-e}, \frac{a+c+e}{1-b-d}\}, & d \geqslant e, \\ \max\{\frac{a+c+d+e}{1-b-e}, \frac{a+c+e}{1-b-d-e}\}, & d < e. \end{cases}$$

It is clear that $\lambda \in [0, 1)$. On the other hand, since

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leqslant p(x_n, x_{n+1}),$$

then we have

$$\max\{p(x_n, x_n), p(x_{n+1}, x_{n+1})\} \leqslant \lambda^n p(x_1, x_0). \tag{2.10}$$

Therefore

$$\begin{aligned} p^s(x_n, x_{n+1}) &= 2p(x_n, x_{n+1}) - p(x_n, x_n) - p(x_{n+1}, x_{n+1}) \\ &\leqslant 2p(x_n, x_{n+1}) + p(x_n, x_n) + p(x_{n+1}, x_{n+1}) \\ &\leqslant 4\lambda^n p(x_1, x_0). \end{aligned}$$

This shows that $\lim_{n\to\infty} p^s(x_n, x_{n+1}) = 0$. Now we have

$$\begin{aligned} p^s(x_{n+k}, x_n) &\leqslant p^s(x_{n+k}, x_{n+k-1}) + \cdots + p^s(x_{n+1}, x_n) \\ &\leqslant 4\lambda^{n+k-1} p(x_1, x_0) + \cdots + 4\lambda^n p(x_1, x_0) \\ &= 4\frac{\lambda^n(1-\lambda^k)}{1-\lambda} p(x_1, x_0) \\ &\leqslant 4\frac{\lambda^n}{1-\lambda} p(x_1, x_0). \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s). Since (X, p) is complete then from Lemma 1, the sequence $\{x_n\}$ is converges in the metric space (X, p^s), say $\lim_{n\to\infty} p^s(x_n, x) = 0$. Again from Lemma 1, we have

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{2.11}$$

Moreover since $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s), we have $\lim_{n,m\to\infty} p^s(x_n, x_m) = 0$ and from (2.10) we have $\lim_{n\to\infty} p(x_n, x_n) = 0$, thus from the definition p^s we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore from (2.11) we have $p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m) = 0$. Now we show that $p(x, Fx) = 0$. Assume this is not true, then from (2.6) we obtain

$$\begin{aligned} p(x, Fx) &\leqslant p(x, Fx_n) + p(Fx_n, Fx) - p(Fx_n, Fx_n) \\ &\leqslant p(x, x_{n+1}) + p(Fx_n, Fx) \\ &\leqslant p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}), dp(x, x_{n+1}) + ep(x_n, Fx) \\ &\leqslant p(x, x_{n+1}) + ap(x, x_n) + bp(x, Fx) + cp(x_n, x_{n+1}), dp(x, x_{n+1}) + ep(x_n, x) + ep(x, Fx) \end{aligned}$$

letting $n \to \infty$, we have

$$p(x, Fx) \leqslant (b + e)p(x, Fx),$$

which is a contradiction. Thus $p(x, Fx) = 0$ and so $x = Fx$. The uniqueness of fixed point follows from (2.6) easily. □

We can have the following corollaries from Theorem 2.

Corollary 2 *(Banach type). Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant \alpha p(x, y)$$

for all $x, y \in X$, where $0 \leqslant \alpha < 1$. Then F has a unique fixed point.

Corollary 3 *(Kannan type). Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant \beta p(x, Fx) + \gamma p(y, Fy)$$

for all $x, y \in X$, where $\beta, \gamma \geqslant 0$ and $\beta + \gamma < 1$. Then F has a unique fixed point.

Corollary 4 *(Reich type). Let (X, p) be a complete partial metric space and let $F : X \to X$ be a map such that*

$$p(Fx, Fy) \leqslant \alpha p(x, y) + \beta p(x, Fx) + \gamma p(y, Fy)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma \geqslant 0$ and $\alpha + \beta + \gamma < 1$. Then F has a unique fixed point.

3. A homotopy result

Now we state a homotopy result.

Theorem 3. *Let (X, p) be a complete partial metric space, U an open subset of X. Suppose that $H : \overline{U} \times [0,1] \to X$ with the following properties*:

1. $x \neq H(x,\lambda)$ *for every* $x \in \partial U$ *and* $\lambda \in [0,1]$ (*here* ∂U *denotes the boundary of* U *in* X).
2. *For all* $x, y \in \overline{U}$ *and* $\lambda \in [0,1]$, $L \in [0,1)$, *such that*

$$p\big(H(x,\lambda), H(y,\lambda)\big) \leqslant Lp(x,y).$$

3. *There exists* $M \geqslant 0$, *such that*

$$p\big(H(x,\lambda), H(x,\mu)\big) \leqslant M|\lambda-\mu|$$

for every $x \in \overline{U}$ *and* $\lambda, \mu \in [0,1]$.

If $H(\cdot,0)$ *has a fixed point in* U, *then* $H(\cdot,1)$ *has a fixed point in* U.

Proof. Consider the set

$$A = \big\{\lambda \in [0,1]\colon\ x = H(x,\lambda) \text{ for some } x \in U\big\}$$

Since $H(\cdot,0)$ has a fixed point in U, then A is nonempty, that is, $0 \in A$. We will show that A is both open and closed in $[0,1]$ and hence by connectedness we have that $A = [0,1]$. As a result, $H(\cdot,1)$ has a fixed point in U. We first show that A is closed in $[0,1]$. To see this let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1,2,3,\ldots,$ there exists $x_n \in U$ with $x_n = H(x_n,\lambda_n)$. Also for $n, m \in \{1,2,3,\ldots\}$ we have

$$\begin{aligned}
p(x_n,x_m) &= p\big(H(x_n,\lambda_n), H(x_m,\lambda_m)\big)\\
&\leqslant p\big(H(x_n,\lambda_n), H(x_n,\lambda_m)\big) + p\big(H(x_n,\lambda_m), H(x_m,\lambda_m)\big) - p\big(H(x_n,\lambda_m), H(x_n,\lambda_m)\big)\\
&\leqslant p\big(H(x_n,\lambda_n), H(x_n,\lambda_m)\big) + p\big(H(x_n,\lambda_m), H(x_m,\lambda_m)\big)\\
&\leqslant M|\lambda_n-\lambda_m| + Lp(x_n,x_m),
\end{aligned}$$

that is,

$$p(x_n,x_m) \leqslant \left(\frac{M}{1-L}\right)|\lambda_n-\lambda_m|.$$

Since $\{\lambda_n\}_{n=1}^{\infty}$ is a Cauchy sequence we have that $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, that is, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete there exists $x \in \overline{U}$ with $p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) = 0$. Since

$$\begin{aligned}
p\big(x_n, H(x,\lambda)\big) &= p\big(H(x_n,\lambda_n), H(x,\lambda)\big)\\
&\leqslant p\big(H(x_n,\lambda_n), H(x_n,\lambda)\big) + p\big(H(x_n,\lambda), H(x,\lambda)\big) - p\big(H(x_n,\lambda), H(x_n,\lambda)\big)\\
&\leqslant p\big(H(x_n,\lambda_n), H(x_n,\lambda)\big) + p\big(H(x_n,\lambda), H(x,\lambda)\big)\\
&\leqslant M|\lambda_n-\lambda| + Lp(x_n,x),
\end{aligned}$$

we have $\lim_{n\to\infty} p(x_n, H(x,\lambda)) = 0$ and so

$$\lim_{n\to\infty} p\big(x_n, H(x,\lambda)\big) = p\big(x, H(x,\lambda)\big) = 0.$$

Thus $\lambda \in A$ and A is closed in $[0,1]$.

Next we show that A is an open in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0,\lambda_0)$. Since U is open, then there exists $r > 0$ such that $B_p(x_0,r) \subseteq U$. Now, let $\delta = p(x_0,\partial U) = \inf\{p(x_0,x)\colon\ x \in \partial U\}$. Then from Lemma 2, $r = \delta - p(x_0,x_0) > 0$. Fix $\varepsilon > 0$ with

$$\varepsilon < \frac{(1-L)\delta}{M}.$$

Let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, then for $x \in \overline{B_p(x_0, r)} = \{x \in X\colon\ p(x, x_0) \leqslant r + p(x_0, x_0)\}$,

$$\begin{aligned} p\big(H(x,\lambda), x_0\big) &= p\big(H(x,\lambda), H(x_0,\lambda_0)\big) \\ &\leqslant p\big(H(x,\lambda), H(x,\lambda_0)\big) + p\big(H(x,\lambda_0), H(x_0,\lambda_0)\big) - p\big(H(x,\lambda_0), H(x,\lambda_0)\big) \\ &\leqslant p\big(H(x,\lambda), H(x,\lambda_0)\big) + p\big(H(x,\lambda_0), H(x_0,\lambda_0)\big) \\ &\leqslant M|\lambda - \lambda_0| + Lp(x, x_0) \\ &\leqslant (1-L)\delta + L\big(r + p(x_0, x_0)\big) \\ &= r + p(x_0, x_0). \end{aligned}$$

Thus for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$,

$$H(\cdot, \lambda) : \overline{B_p(x_0, r)} \to \overline{B_p(x_0, r)}.$$

We can now apply Corollary 2 to deduce that $H(\cdot, \lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in U since (1) holds. Thus $\lambda \in A$ for any $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ and therefore A is open in $[0, 1]$. $\square$

Acknowledgements

The authors thank the referees for their appreciation, valuable comments and suggestions.

References

[1] R.P. Agarwal, D. O'Regan, M. Sambandham, Random and deterministic fixed point theory for generalized contractive maps, Appl. Anal. 83 (2004) 711–725.
[2] D.W. Boyd, J.S.W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc. 20 (2) (1969) 458–465.
[3] Lj.B. Ćirić, Fixed points for generalized multi-valued mappings, Mat. Vesnik 9 (24) (1972) 265–272.
[4] Lj.B. Ćirić, Generalized contractions and fixed point theorems, Publ. Inst. Math. 12 (26) (1971) 19–26.
[5] M.H. Escardo, PCF extended with real numbers, Theoret. Comput. Sci. 162 (1996) 79–115.
[6] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Canad. Math. Bull. 16 (1973) 201–206.
[7] R. Heckmann, Approximation of metric spaces by partial metric spaces, Appl. Categ. Structures 7 (1999) 71–83.
[8] R. Kannan, Some results on fixed points, Bull. Calcutta Math. Soc. 60 (1968) 71–76.
[9] R. Kannan, Some results on fixed points-II, Amer. Math. Monthly 76 (1969) 405–408.
[10] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, Proc. Amer. Math. Soc. 62 (2) (1977) 344–348.
[11] S.G. Matthews, Partial metric topology, in: Proc. 8th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 728, 1994, pp. 183–197.
[12] S. Oltra, O. Valero, Banach's fixed point theorem for partial metric spaces, Rend. Istit. Mat. Univ. Trieste 36 (2004) 17–26.
[13] S.J. O'Neill, Partial metrics, valuations and domain theory, in: Proc. 11th Summer Conference on General Topology and Applications, in: Ann. New York Acad. Sci., vol. 806, 1996, pp. 304–315.
[14] S. Reich, Kannan's fixed point theorem, Boll. Unione Mat. Ital. 4 (4) (1971) 1–11.
[15] S. Romaguera, A Kirk type characterization of completeness for partial metric spaces, Fixed Point Theory Appl. (2010), Article ID 493298, 6 pp.
[16] O. Valero, On Banach fixed point theorems for partial metric spaces, Appl. Gen. Topol. 6 (2) (2005) 229–240.